1. (12 points) Find the area of the region bounded by the curves $y = 2 - x$, $y = 4 - x^2$, and $x = 5$. **You do not have to evaluate the integral**.

   First, **draw the picture!**. Then find the intersection points for the curves $y = 2 - x$ and $y = 4 - x^2$:

$$\begin{aligned} 2 - x &= 4 - x^2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ \text{Hence, } x &= 2, -1. \end{aligned}$$

   Looking at the picture you will see two regions that are bounded by these three curves and the area is given by

$$\text{Area } = \int_{-1}^{2} [(4 - x^2) - (x - 2)]dx + \int_{2}^{5} [(x - 2) - (4 - x^2)]dx.$$

2. (12 points) Find the area of the region bounded by the curves $y = x - 2$, $y = 4 - x^2$, and $x = 4$. **You do not have to evaluate the integral**.

   First, **draw the picture!**. Then find the intersection points for the curves $y = 2 - x$ and $y = 4 - x^2$:

$$\begin{aligned} x - 2 &= 4 - x^2 \\ x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \\ \text{Hence, } x &= -3, 2. \end{aligned}$$

   Looking at the picture you will see two regions that are bounded by these three curves and the area is given by

$$\text{Area } = \int_{-3}^{2} [(4 - x^2) - (x - 2)]dx + \int_{2}^{4} [(x - 2) - (4 - x^2)]dx.$$

3. (12 points) Find the area of the region bounded by the curves $y = 2 - x$, $y = 4 - x^2$, and $x = -5$. **You do not have to evaluate the integral**.

First, **draw the picture!**. Then find the intersection points for the curves $y = 2 - x$ and $y = 4 - x^2$:

$$\begin{aligned} 2 - x &= 4 - x^2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ \text{Hence, } x &= 2, -1. \end{aligned}$$

Looking at the picture you will see two regions that are bounded by these three curves and the area is given by

$$\text{Area } = \int_{-5}^{-1} [(x-2) - (4-x^2)]dx + \int_{-1}^{2} [(4-x^2) - (x-2)]dx.$$

4. (12 points) Find the area of the region bounded by the curves $y = x - 3$, $y = 9 - x^2$, and $x = 4$. **You do not have to evaluate the integral**.

First, **draw the picture!**. Then find the intersection points for the curves $y = x - 3$ and $y = 9 - x^2$:

$$\begin{aligned} x - 3 &= 9 - x^2 \\ x^2 + x - 12 &= 0 \\ (x+4)(x-3) &= 0 \\ \text{Hence, } x &= -4, 3. \end{aligned}$$

Looking at the picture you will see two regions that are bounded by these three curves and the area is given by

$$\text{Area } = \int_{-4}^{3} [(9-x^2) - (x-3)]dx + \int_{3}^{4} [(x-3) - (9-x^2)]dx.$$

5. (12 points) Find the area of the region bounded by the curves $x + y = 0$ and $x = y^2 + 3y$.

   Note that since the second equation is difficult to solve for y, we will write our equations as functions of y and therefore integrate with respect to y.

   Next, **draw the picture!**. Then find the intersection points for the curves $x = -y$ and $x = y^2 + 3y$:

$$\begin{aligned} -y &= y^2 + 3y \\ y^2 + 4y &= 0 \\ y(y+4) &= 0 \\ \text{Hence, } y &= -4, 0. \end{aligned}$$

   Looking at the picture you will see that the area is given by

$$\begin{aligned} \text{Area } &= \int_{-4}^{0} [(-y) - (y^2 + 3y)]dy \\ &= \int_{-4}^{0} [-y^2 - 4y)]dy \\ &= \int_{0}^{-4} [y^2 + 4y)]dy \\ &= \left(\frac{1}{3}y^3 + 2y^2\right)_0^{-4} \\ &= \left(-\frac{64}{3} + 32\right) - 0 \\ &= \frac{32}{3}. \end{aligned}$$

6. (12 points)Find the volume of the solid obtained by rotating the region bounded $y = x$ and $y = x^2$ about the x-axis. **You do not have to evaluate the integral**.

Draw the picture!!!

The intersection points are:

$$\begin{aligned} x &= x^2 \\ x^2 - x &= 0 \\ x(x-1) &= 0 \\ \text{Hence, } x &= 0, 1. \end{aligned}$$

Washer Method
If you chose to slice perpendicular to the x-axis, you must use the *washer method* where your washer has thickness dx:

$$\begin{aligned} \text{Volume}_{\text{washer}} &= \pi(r_{outer}^2 - r_{inner}^2)dx \\ &= \pi[x^2 - (x^2)^2]dx \\ &= \pi[x^2 - x^4]dx. \end{aligned}$$

Therefore,

$$\text{Volume}_{\text{solid}} = \pi \int_0^1 [x^2 - x^4]dx = \frac{2\pi}{15}.$$

Shell Method
If you chose to slice parallel to the x-axis, you must use the *shell method* where your shell has thickness dy (and thus you are working with the equations $x = y$ and $x = \sqrt{y}$ and the intersection points 0 and 1):

$$\begin{aligned} \text{Volume}_{\text{shell}} &= l \cdot h \cdot dy \\ &= 2\pi y \cdot (\sqrt{y} - y)dy \\ &= 2\pi(y^{\frac{3}{2}} - y^2)dy. \end{aligned}$$

Therefore,

$$\text{Volume}_{\text{solid}} = 2\pi \int_0^1 (y^{\frac{3}{2}} - y^2)dy = \frac{2\pi}{15}.$$

7. (12 points)Find the volume of the solid obtained by rotating the region bounded $y = x$ and $y = x^4$ about the x-axis. **You do not have to evaluate the integral**.

Draw the picture!!!

The intersection points are:

$$\begin{aligned} x &= x^4 \\ x^4 - x &= 0 \\ x(x^3 - 1) &= 0 \end{aligned}$$

Hence, $x = 0, 1,$ and two complex roots.

Washer Method
If you chose to slice perpendicular to the x-axis, you must use the *washer method* where your washer has thickness dx:

$$\begin{aligned} \text{Volume}_{\text{washer}} &= \pi(r_{outer}^2 - r_{inner}^2)dx \\ &= \pi[x^2 - (x^4)^2]dx \\ &= \pi[x^2 - x^8]dx. \end{aligned}$$

Therefore,

$$\text{Volume}_{\text{solid}} = \pi \int_0^1 [x^2 - x^8]dx = \frac{2\pi}{9}.$$

Shell Method
If you chose to slice parallel to the x-axis, you must use the *shell method* where your shell has thickness dy (and thus you are working with the equations $x = y$ and $x = \sqrt[4]{y}$ and the intersection points 0 and 1):

$$\begin{aligned} \text{Volume}_{\text{shell}} &= l \cdot h \cdot dy \\ &= 2\pi y \cdot (\sqrt[4]{y} - y)dy \\ &= 2\pi(y^{\frac{5}{4}} - y^2)dy. \end{aligned}$$

Therefore,

$$\text{Volume}_{\text{solid}} = 2\pi \int_0^1 (y^{\frac{5}{4}} - y^2)dy = \frac{2\pi}{9}.$$

8. (12 points)Find the volume of the solid obtained by rotating the region bounded $y = x$ and $y = x^4$ about the y-axis. **You do not have to evaluate the integral**.

Draw the picture!!!

The intersection points are:

$$\begin{aligned} x &= x^4 \\ x^4 - x &= 0 \\ x(x^3 - 1) &= 0 \\ &\text{Hence, } x = 0, 1, \text{and two complex roots.} \end{aligned}$$

Washer Method
If you chose to slice perpendicular to the y-axis, you must use the *washer method* where your washer has thickness dy (and thus you are working with the equations $x = y$ and $x = \sqrt[4]{y}$ and the intersection points 0 and 1):

$$\begin{aligned} \text{Volume}_{\text{washer}} &= \pi(r_{outer}^2 - r_{inner}^2)dy \\ &= \pi[\sqrt[4]{y}^2 - y^2]dy \\ &= \pi[y^{\frac{1}{2}} - y^2]dy. \end{aligned}$$

Therefore,

$$\text{Volume}_{\text{solid}} = \pi \int_0^1 [y^{\frac{1}{2}} - y^2]dy = \frac{\pi}{3}.$$

Shell Method
If you chose to slice parallel to the y-axis, you must use the *shell method* where your shell has thickness dx :

$$\begin{aligned} \text{Volume}_{\text{shell}} &= l \cdot h \cdot dx \\ &= 2\pi x \cdot (x - x^4)dx \\ &= 2\pi(x^2 - x^5)dx. \end{aligned}$$

Therefore,

$$\text{Volume}_{\text{solid}} = 2\pi \int_0^1 (x^2 - x^5)dx = \frac{\pi}{3}.$$

9. (12 points)Find the volume of the solid obtained by rotating the region bounded $y = x$ and $y = x^3$ about the x-axis. **You do not have to evaluate the integral**.

Draw the picture!!!

The intersection points are:

$$\begin{aligned} x &= x^3 \\ x^3 - x &= 0 \\ x(x^2-1) &= 0 \\ \text{Hence, } x &= 0, -1, 1. \end{aligned}$$

Washer Method
If you chose to slice perpendicular to the x-axis, you must use the *washer method* where your washer has thickness dx:

$$\begin{aligned} \text{Volume}_{\text{washer}} &= \pi(r_{outer}^2 - r_{inner}^2)dx \\ &= \pi[x^2 - (x^3)^2]dx \\ &= \pi[x^2 - x^6]dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Volume}_{\text{solid}} &= \pi \int_{-1}^{1} [x^2 - x^6]dx \\ &= 2\pi \int_{0}^{1} [x^2 - x^6]dx = \frac{8\pi}{21}. \end{aligned}$$

Shell Method
If you chose to slice parallel to the x-axis, you must use the *shell method* where your shell has thickness dy (and thus you are working with the equations $x = y$ and $x = \sqrt[3]{y}$ and the intersection points -1, 0 and 1):

$$\begin{aligned} \text{Volume}_{\text{shell}} &= l \cdot h \cdot dy \\ &= 2\pi y \cdot (\sqrt[3]{y} - y)dy \\ &= 2\pi(y^{\frac{4}{3}} - y^2)dy. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Volume}_{\text{solid}} &= 2\pi \int_{-1}^{1} (y^{\frac{4}{3}} - y^2)dy \\ &= 4\pi \int_{0}^{1} (y^{\frac{4}{3}} - y^2)dy = \frac{8\pi}{21}. \end{aligned}$$

10. (12 points)Find the volume of the solid obtained by rotating the region bounded $y = x$ and $y = x^3$ about the y-axis. **You do not have to evaluate the integral**.

Draw the picture!!!

The intersection points are:

$$\begin{aligned} x &= x^3 \\ x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ \text{Hence, } x &= 0, -1, 1. \end{aligned}$$

Washer Method
If you chose to slice perpendicular to the y-axis, you must use the *washer method* where your washer has thickness dy (and thus you are working with the equations $x = y$ and $x = \sqrt[3]{y}$ and the intersection points -1, 0 and 1):

$$\begin{aligned} \text{Volume}_{\text{washer}} &= \pi(r_{outer}^2 - r_{inner}^2)dy \\ &= \pi[\sqrt[3]{y}^2 - y^2]dy \\ &= \pi[y^{\frac{2}{3}} - y^2]dy. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Volume}_{\text{solid}} &= \pi \int_{-1}^{1} [y^{\frac{2}{3}} - y^2]dy \\ &= 2\pi \int_{0}^{1} [y^{\frac{2}{3}} - y^2]dy = \frac{8\pi}{15}. \end{aligned}$$

Shell Method
If you chose to slice parallel to the y-axis, you must use the *shell method* where your shell has thickness dx :

$$\begin{aligned} \text{Volume}_{\text{shell}} &= l \cdot h \cdot dx \\ &= 2\pi x \cdot (x - x^3)dx \\ &= 2\pi(x^2 - x^4)dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Volume}_{\text{solid}} &= 2\pi \int_{-1}^{1} (x^2 - x^4)dx \\ &= 4\pi \int_{0}^{1} (x^2 - x^4)dx = \frac{8\pi}{15}. \end{aligned}$$

11. (14 points)Find the volume of the solid obtained by rotating the region bounded $y = x$ and $y = x^2$ about the line $y = 2$.

Draw the picture!!!

The intersection points are:

$$\begin{aligned} x &= x^2 \\ x^2 - x &= 0 \\ x(x-1) &= 0 \\ \text{Hence, } x &= 0, 1. \end{aligned}$$

Washer Method
If you chose to slice perpendicular to the line $y = 2$, you must use the *washer method* where your washer has thickness dx:

$$\begin{aligned} \text{Volume}_{\text{washer}} &= \pi(r_{outer}^2 - r_{inner}^2)dx \\ &= \pi[(2-x^2)^2 - (2-x)^2]dx \\ &= \pi[x^4 - 5x^2 + 4x]dx. \end{aligned}$$

Therefore,

$$\text{Volume}_{\text{solid}} = \pi \int_0^1 [x^4 - 5x^2 + 4x]dx = \frac{8\pi}{15}.$$

Shell Method
If you chose to slice parallel to the line $y = 2$, you must use the *shell method* where your shell has thickness dy (and thus you are working with the equations $x = y$ and $x = \sqrt{y}$ and the intersection points 0 and 1):

$$\begin{aligned} \text{Volume}_{\text{shell}} &= l \cdot h \cdot dy \\ &= 2\pi(2-y) \cdot (\sqrt{y} - y)dy \\ &= 2\pi(2\sqrt{y} - 2y - y^{\frac{3}{2}} + y^2)dy. \end{aligned}$$

Therefore,

$$\text{Volume}_{\text{solid}} = 2\pi \int_0^1 (2\sqrt{y} - 2y - y^{\frac{3}{2}} + y^2)dy = \frac{8\pi}{15}.$$

12. (12 points)A tank has the shape of the bottom half of a sphere of radius $5m$ It is filled with water to a height of $3m$. Find the work required to empty the tank by pumping all the water out the top of the tank (the density of water is $1000kg/m^3$, mass = density · volume and $g = 9.8m/s^2$). **You do not have to evaluate the integral**.

13. (12 points)A tank has the shape of the bottom half of a sphere of radius $6m$ It is filled with water to a height of $4m$. Find the work required to empty the tank by pumping all the water out the top of the tank (the density of water is $1000kg/m^3$, mass = density · volume and $g = 9.8m/s^2$). **You do not have to evaluate the integral**.

14. (12 points)A tank has the shape of the bottom half of a sphere of radius $5m$ It is filled with water to a height of $4m$. Find the work required to empty the tank by pumping all the water out the top of the tank (the density of water is $1000kg/m^3$, mass = density · volume and $g = 9.8m/s^2$). **You do not have to evaluate the integral**.

15. (12 points)A tank has the shape of an inverted circular cone with height $10m$ and base radius $5m$. It is filled with water to a height of $7m$. Find the work required to empty the tank by pumping all the water out the top of the tank (the density of water is $1000kg/m^3$, mass = density · volume and $g = 9.8m/s^2$). **You do not have to evaluate the integral**.

16. (12 points)A tank has the shape of an inverted circular cone with height $10m$ and base radius $5m$. It is filled with water to a height of $8m$. Find the work required to empty the tank by pumping all the water out the top of the tank (the density of water is $1000kg/m^3$, mass = density · volume and $g = 9.8m/s^2$). **You do not have to evaluate the integral**.

17. (12 points)A tank has the shape of an inverted circular cone with height $10m$ and base radius $4m$. It is filled with water to a height of $7m$. Find the work required to empty the tank by pumping all the water out the top of the tank (the density of water is $1000kg/m^3$, mass = density · volume and $g = 9.8m/s^2$). **You do not have to evaluate the integral**.

18. (10 points) Solve the differential equation $\frac{dy}{dx} = \frac{y \cos x}{1+y^2}$ subject to $y = 1$ when $x = 0$. You do not have to solve for y.

This is a separable equation that can be rewritten to

$$\frac{1+y^2}{y} dy = \cos x dx.$$

This can be written as

$$\left(\frac{1}{y} + y\right) dy = \cos x dx.$$

Integrating both sides yields

$$\ln|y| + \frac{1}{2}y^2 = \sin x + C.$$

Using $y(0) = 1$,

$$C = \ln|y| + \frac{1}{2}y^2 - \sin x = \ln 1 + \frac{1}{2} - \sin 0 = \frac{1}{2}.$$

Hence our answer is

$$\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}.$$

19. (10 points) Solve the differential equation $\frac{dy}{dx} = \frac{y \sin x}{1+y^2}$ subject to $y = 1$ when $x = 0$. You do not have to solve for y.

This is a separable equation that can be rewritten to

$$\frac{1+y^2}{y} dy = \sin x dx.$$

This can be written as

$$\left(\frac{1}{y} + y\right) dy = \sin x dx.$$

Integrating both sides yields

$$\ln|y| + \frac{1}{2}y^2 = -\cos x + C.$$

Using $y(0) = 1$,

$$C = \ln|y| + \frac{1}{2}y^2 + \cos x = \ln 1 + \frac{1}{2} + \cos 0 = \frac{3}{2}.$$

Hence our answer is

$$\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{3}{2}.$$

20. (10 points) Solve the differential equation $\frac{dy}{dx} = \frac{y\cos x}{1+y^3}$ subject to $y = 1$ when $x = 0$. You do not have to solve for y.

This is a separable equation that can be rewritten to

$$\frac{1+y^3}{y}dy = \cos x dx.$$

This can be written as

$$\left(\frac{1}{y} + y^2\right) dy = \cos x dx.$$

Integrating both sides yields

$$\ln|y| + \frac{1}{3}y^3 = \sin x + C.$$

Using $y(0) = 1$,

$$C = \ln|y| + \frac{1}{3}y^3 - \sin x = \ln 1 + \frac{1}{3} - \sin 0 = \frac{1}{3}.$$

Hence our answer is

$$\ln|y| + \frac{1}{3}y^3 = \sin x + \frac{1}{3}.$$

21. (10 points) What is the equation of the curve that passes through the point $(1, 1)$ and whose slope at (x,y) is $\frac{y^2}{x^3}$? (So $\frac{dy}{dx} = \frac{y^2}{x^3}$.)

This is a separable equation that can be rewritten to

$$\frac{1}{y^2}dy = \frac{1}{x^3}dx.$$

Integrating both sides yields

$$-\frac{1}{y} = -\frac{1}{2x^2} + C.$$

Using the fact that we are at the point $(1, 1)$,

$$C = -\frac{1}{y} + \frac{1}{2x^2} = -1 + \frac{1}{2} = -\frac{1}{2}.$$

Hence we have

$$-\frac{1}{y} = -\frac{1}{2x^2} + -\frac{1}{2} \Leftrightarrow \frac{1}{y} = \frac{1}{2x^2} + \frac{1}{2} = \frac{1+x^2}{2x^2}.$$

Thus

$$y = \frac{2x^2}{1+x^2}.$$

Evaluate each of the following integrals (if they converge):

22. (9 points)

$$\int \sin^3 x \cos^6 x dx = \int \sin^2 x \cos^6 x \sin x dx$$
$$= \int (1 - \cos^2 x) \cos^6 x \sin x dx = (*)$$

Let $u = \cos x$. Then $du = -\sin x dx$. Hence

$$(*) = -\int [(1 - u^2)u^6] du = -\int [u^6 - u^8] du$$
$$= \frac{u^9}{9} - \frac{u^7}{7} + C = \frac{\cos^9 x}{9} - \frac{\cos^7 x}{7} + C$$

23. (9 points)

$$\int \sin^3 x \cos^4 x dx = \int \sin^2 x \cos^4 x \sin x dx$$
$$= \int (1 - \cos^2 x) \cos^4 x \sin x dx = (**)$$

Let $u = \cos x$. Then $du = -\sin x dx$. Hence

$$(**) = -\int [(1 - u^2)u^4] du = -\int [u^4 - u^6] du$$
$$= \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$$

24. (9 points)

$$\int \sin^4 x \cos^3 x dx = \int \sin^4 x \cos^2 x \cos x dx$$
$$= \int \sin^4 x (1 - \sin^2 x) \cos x dx = (***)$$

Let $u = \sin x$. Then $du = \cos x dx$. Hence

$$(***) = \int u^4 [(1 - u^2)] du = \int [u^4 - u^6] du$$
$$= \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$$

25. (9 points) $\int \sin^3 x \cos^3 x dx$. There are two ways to do this problem:

$$\begin{aligned}\int \sin^3 x \cos^3 x dx &= \int \sin^3 x \cos^2 x \cos x dx \\ &= \int \sin^3 x(1-\sin^2 x)\cos x dx = (*)\end{aligned}$$

Let $u = \sin x$. Then $du = \cos x dx$. Hence

$$\begin{aligned}(*) &= \int u^3[(1-u^2)]du = \int [u^3 - u^5]du \\ &= \frac{u^4}{4} - \frac{u^6}{6} + C = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C\end{aligned}$$

or

$$\begin{aligned}\int \sin^3 x \cos^3 x dx &= \int \sin^2 x \cos^3 x \sin x dx \\ &= \int (1-\cos^2 x)\cos^3 x \sin x dx = (**)\end{aligned}$$

Let $u = \cos x$. Then $du = -\sin x dx$. Hence

$$\begin{aligned}(**) &= -\int [(1-u^2)u^3]du = -\int [u^3 - u^5]du \\ &= \frac{u^6}{6} - \frac{u^4}{4} + C = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C\end{aligned}$$

26. (9 points) $\int t \sin 2t dt$.

Let $u = t$ and $dv = \sin 2t dt$.

Then $du = dt$ and $v = -\frac{1}{2}\cos 2t$.

Hence

$$\int t \sin 2t dt = -\frac{t}{2}\cos 2t + \frac{1}{2}\int \cos 2t dt = -\frac{t}{2}\cos 2t + \frac{1}{4}\sin 2t + C.$$

27. (9 points) $\int t \cos 2t dt$.

Let $u = t$ and $dv = \cos 2t dt$.

Then $du = dt$ and $v = \frac{1}{2}\sin 2t$.

Hence

$$\int t \sin 2t dt = \frac{t}{2}\sin 2t - \frac{1}{2}\int \sin 2t dt = \frac{t}{2}\sin 2t + \frac{1}{4}\cos 2t + C.$$

28. (9 points)

$$\begin{aligned}\int_{-\infty}^{-1} e^{-3t}dt &:= \lim_{s\to-\infty}\int_{s}^{-1} e^{-3t}dt\\ &= \lim_{s\to-\infty}\left(-\frac{1}{3}e^{-3t}\right)_{s}^{-1}\\ &= \lim_{s\to-\infty}\left(-\frac{1}{3}e^{3}+\frac{1}{3}e^{-3s}\right)\\ &= -\frac{1}{3}e^{3}+\frac{1}{3}e^{\infty},\ \text{i.e., the integral diverges.}\end{aligned}$$

29. (9 points)

$$\begin{aligned}\int_{-\infty}^{-1} e^{2t}dt &:= \lim_{s\to-\infty}\int_{s}^{-1} e^{2t}dt\\ &= \lim_{s\to-\infty}\left(\frac{1}{2}e^{2t}\right)_{s}^{-1}\\ &= \lim_{s\to-\infty}\left(\frac{1}{2}e^{-2}-\frac{1}{2}e^{2s}\right)\\ &= \frac{1}{2e^{2}}-\frac{1}{2}e^{-\infty}=\frac{1}{2e^{2}}\end{aligned}$$

30. (9 points)

$$\begin{aligned}\int_{-\infty}^{-1} e^{-2t}dt &:= \lim_{s\to-\infty}\int_{s}^{-1} e^{-2t}dt\\ &= \lim_{s\to-\infty}\left(-\frac{1}{2}e^{-2t}\right)_{s}^{-1}\\ &= \lim_{s\to-\infty}\left(-\frac{1}{2}e^{2}+\frac{1}{2}e^{-2s}\right)\\ &= -\frac{1}{2}e^{2}+\frac{1}{2}e^{\infty},\ \text{i.e., the integral diverges.}\end{aligned}$$

31. (10 points)

$$\begin{aligned}\int_{1}^{\infty} e^{-2t}dt &:= \lim_{s\to\infty}\int_{1}^{\infty} e^{-2t}dt\\ &= \lim_{s\to\infty}\left(-\frac{1}{2}e^{-2t}\right)_{1}^{s}\\ &= \lim_{s\to\infty}\left(-\frac{1}{2}e^{-2}+\frac{1}{2}e^{-2s}\right)\\ &= -\frac{1}{2e^{2}}+\frac{1}{2}e^{-\infty}=\frac{1}{2e^{2}}\end{aligned}$$

32. (9 points)

$$\begin{aligned}\int_{-5}^{5} \frac{1}{x^4} dx &= 2\int_{0}^{5} \frac{1}{x^4} dx \\ &:= 2 \lim_{t \to 0^+} \int_{t}^{5} \frac{1}{x^4} dx \\ &= 2 \lim_{t \to 0^+} \left(-\frac{1}{3x^3} dx \right)_{t}^{5} \\ &= 2 \lim_{t \to 0^+} \left(\frac{1}{3t^3} - \frac{1}{3 \cdot 5^3} \right), \text{ i.e., the integral diverges.}\end{aligned}$$

33. (9 points)

$$\begin{aligned}\int_{-3}^{3} \frac{1}{x^2} dx &= 2\int_{0}^{3} \frac{1}{x^2} dx \\ &:= 2 \lim_{t \to 0^+} \int_{t}^{3} \frac{1}{x^2} dx \\ &= 2 \lim_{t \to 0^+} \left(-\frac{1}{x} dx \right)_{t}^{3} \\ &= 2 \lim_{t \to 0^+} \left(\frac{1}{t} - \frac{1}{3} \right), \text{ i.e., the integral diverges.}\end{aligned}$$

34. (9 points)

$$\begin{aligned}\int_{-2}^{2} \frac{1}{x^2} dx &= 2\int_{0}^{2} \frac{1}{x^2} dx \\ &:= 2 \lim_{t \to 0^+} \int_{t}^{2} \frac{1}{x^2} dx \\ &= 2 \lim_{t \to 0^+} \left(-\frac{1}{x} dx \right)_{t}^{2} \\ &= 2 \lim_{t \to 0^+} \left(\frac{1}{t} - \frac{1}{2} \right), \text{ i.e., the integral diverges.}\end{aligned}$$

35. (9 points) Let $u = 3 - x$. Then $du = -dx$ and when $x = 2$, $u = 1$ and when $x = 3$, $u = 0$.

$$\begin{aligned} \int_2^3 \frac{1}{\sqrt{3-x}}dx &= \int_1^0 -\frac{1}{\sqrt{u}}du = \int_0^1 \frac{1}{\sqrt{u}}du \\ &:= \lim_{t\to 0^+} \int_t^1 u^{-\frac{1}{2}}du \\ &= \lim_{t\to 0^+} \left(2u^{\frac{1}{2}}\right)_t^1 \\ &= \lim_{t\to 0^+} \left(2 - 2\sqrt{t}\right) = 2. \end{aligned}$$

36. (9 points) Let $u = x - 2$. Then $du = dx$ and when $x = 2$, $u = 0$ and when $x = 3$, $u = 1$.

$$\int_2^3 \frac{1}{\sqrt{x-2}}dx = \int_0^1 \frac{1}{\sqrt{u}}du = \ldots = 2 \text{ by the previous problem.}$$

37. (10 points) Let $u = 3 + x$. Then $du = dx$ and when $x = -3$, $u = 0$ and when $x = -2$, $u = 1$.

$$\int_{-3}^{-2} \frac{1}{\sqrt{3+x}}dx = \int_0^1 \frac{1}{\sqrt{u}}du = \ldots = 2 \text{ by the previous problem.}$$

38. (9 points) $\int \frac{x+8}{x^2+x-6}dx$.

First we note that

$$\frac{x+8}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}.$$

Hence $x + 8 = A(x - 2) + B(x + 3) = Ax - 2A + Bx + 3B$. This gives us

$$\begin{aligned} 1 = A + B &\rightarrow \text{(multiply by 2)} \rightarrow 2 = 2A + 2B \\ 8 = -2A + 3B &\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow 8 = -2A + 3B \end{aligned}$$

Adding the two equations together gives us $10 = 5B$, or $B = 2$. Hence $A = -1$. So

$$\begin{aligned} \int \frac{x+3}{x^2+x-6}dx &= \int \left(\frac{2}{x-2} - \frac{1}{x+3}\right) dx \\ &= 2\ln|x-2| - \ln|x+3| + C. \end{aligned}$$

39. (9 points) $\int \frac{x+7}{x^2-x-2}dx$. First we note that

$$\frac{x+7}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}.$$

Hence $x+7 = A(x+1) + B(x-2) = Ax + A + Bx - 2B$. This gives us

$$\begin{aligned} 1 = A + B &\rightarrow \text{(multiply by 2)} \rightarrow 2 = 2A + 2B \\ 8 = A - 2B &\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow 8 = A - 2B \end{aligned}$$

Adding the two equations together gives us $9 = 3A$, or $A = 3$. Hence $B = -2$. So

$$\begin{aligned} \int \frac{x+7}{x^2-x-2}dx &= \int \left(\frac{3}{x-2} - \frac{2}{x+1} \right) dx \\ &= 3\ln|x-2| - 2\ln|x+1| + C. \end{aligned}$$

40. (9 points) $\int \frac{x+5}{x^2+x-2}dx$. First we note that

$$\frac{x+5}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}.$$

Hence $x+5 = A(x-1) + B(x+2) = Ax - A + Bx + 2B$. This gives us

$$\begin{aligned} 1 &= A + B \\ 5 &= -A + 2B \end{aligned}$$

Adding the two equations together gives us $6 = 3B$, or $B = 2$. Hence $A = -1$. So

$$\begin{aligned} \int \frac{x+5}{x^2+x-2}dx &= \int \left(\frac{2}{x-1} - \frac{1}{x+2} \right) dx \\ &= 2\ln|x-1| - \ln|x+2| + C. \end{aligned}$$

41. (9 points) $\int \frac{x+5}{x^2-x-2}dx$. First we note that

$$\frac{x+5}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}.$$

Hence $x+5 = A(x+1) + B(x-2) = Ax + A + Bx - 2B$. This gives us

$$\begin{aligned} 1 = A + B &\rightarrow \text{(multiply by 2)} \rightarrow 2 = 2A + 2B \\ 5 = A - 2B &\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow 5 = A - 2B \end{aligned}$$

Adding the two equations together gives us $7 = 3A$, or $A = \frac{7}{3}$. Hence $B = -\frac{4}{3}$. So

$$\begin{aligned} \int \frac{x+5}{x^2-x-2}dx &= \int \left(\frac{\frac{7}{3}}{x-2} - \frac{\frac{4}{3}}{x+1} \right) dx \\ &= \frac{7}{3}\ln|x-2| - \frac{4}{3}\ln|x+1| + C. \end{aligned}$$

42. (9 points) Let $x = \sec\theta$. Then $dx = \sec\theta\tan\theta d\theta$. Moreover, when $x = 1$, $1 = \sec\theta \Leftrightarrow 1 = \cos\theta$, i.e. $\theta = 0$. When $x = 2$, $2 = \sec\theta \Leftrightarrow \frac{1}{2} = \cos\theta$, i.e. $\theta = \frac{\pi}{3}$. So

$$\begin{aligned}
\int_1^2 \frac{\sqrt{x^2-1}}{x}dx &= \int_0^{\frac{\pi}{3}} \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta}\sec\theta\tan\theta d\theta \\
&= \int_0^{\frac{\pi}{3}} \sqrt{\tan^2\theta}\tan\theta d\theta \\
&= \int_0^{\frac{\pi}{3}} \tan^2\theta d\theta \\
&= \int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^2\theta}d\theta \\
&= \int_0^{\frac{\pi}{3}} \frac{1-\cos^2\theta}{\cos^2\theta}d\theta \\
&= \int_0^{\frac{\pi}{3}} \left(\sec^2\theta - 1\right) d\theta \\
&= (\tan\theta - \theta)_0^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3}.
\end{aligned}$$

43. (10 points) What is the average value of the curve $y(t) = t\sin 2t$ over the interval $[0, 2\pi]$?

By question 26,

$$\int t\sin 2t dt = -\frac{t}{2}\cos 2t + \frac{1}{4}\sin 2t + C.$$

Hence

$$\begin{aligned}
f_{avg} &= \frac{1}{2\pi - 0}\int_0^{2\pi} t\sin 2t dt \\
&= \frac{1}{2\pi}\left(-\frac{t}{2}\cos 2t + \frac{1}{4}\sin 2t\right)_0^{2\pi} \\
&= \frac{1}{2\pi}\left(\left[-\frac{0}{2}\cos 0 + \frac{1}{4}\sin 0\right] - \left[-\frac{2\pi}{2}\cos 4\pi + \frac{1}{4}\sin 4\pi\right]\right) \\
&= \frac{1}{2\pi}(0 + 0 + \pi - 0) = \frac{1}{2}.
\end{aligned}$$