Key - Wheeler Fall 2008   Calculus II-Exam 2

Answer each of the following. Choose One: Grade my WORK/ANSWERS). If you want your work graded (i.e. if you want partial credit), please show all work (**no work=no credit**).

1. (16 points) Solve the initial-value problem

$$y' = x - y, \qquad y(1) = 1.$$

Solution:
First, rewrite the equation as $y' + y = x$. Then
$I(x) = e^{\int 1dx} = e^x$. Multiply both sides of the equation by this factor to get:

$$e^x y' + ye^x = xe^x.$$

This can be written as

$$\frac{d}{dx}(e^x y) = xe^x.$$

Hence

$$\begin{aligned} e^x y &= \int xe^x dx \quad (\text{Let } u = x \text{ and } dv = e^x dx. \text{ Then } du = dx \text{ and } v = e^x.) \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C. \end{aligned}$$

Dividing both sides by e^x:

$$y = x - 1 + Ce^{-x}.$$

Using the initial value $y(1) = 1$:
$1 = 1 - 1 + Ce^{-1}$ which gives us $C = e$. Hence

$$y = x - 1 + ee^{-x} = x - 1 + e^{1-x}.$$

Alternate Solution:
The complementary equation of $y' + y = x$ is $r + 1 = 0$ which gives $r = -1$. Hence $y_c(x) = c_1 e^{-x}$. As well, $y_p(x) = Ax + B$ thus $y_p'(x) = A$. Substituting these into the equation gives $A + Ax + B = x$ which gives rise (by comparing coefficients) to the equations $A = 1$ and $B + A = 0$. Thus $A = 1$ and $B = -1$, i.e. $y_p(x) = x - 1$. Hence $y(x) = y_c(x) + y_p(x) = c_1 e^{-x} + x - 1$. This leads to the same solution as above.

2. (16 points) Solve the initial-value problem

$$y' = x + y, \qquad y(-1) = 1.$$

Solution:
First, rewrite the equation as $y' - y = x$. Then
$I(x) = e^{\int -1 dx} = e^{-x}$. Multiply both sides of the equation by this factor to get:

$$e^{-x}y' - ye^{-x} = xe^{-x}.$$

This can be written as

$$\frac{d}{dx}(e^{-x}y) = xe^{-x}.$$

Hence

$$\begin{aligned} e^{-x}y &= \int xe^{-x}dx \quad \text{(Let } u = x \text{ and } dv = e^{-x}dx\text{. Then } du = dx \text{ and } v = -e^{-x}\text{.)} \\ &= -xe^{-x} + \int e^{-x}dx \\ &= -xe^{-x} - e^{-x} + C. \end{aligned}$$

Multiplying both sides by e^x:

$$y = -x - 1 + Ce^x.$$

Using the initial value $y(-1) = 1$:
$1 = 1 - 1 + Ce^{-1}$ which gives us $C = e$. Hence

$$y = -x - 1 + ee^x = e^{x+1} - x - 1.$$

Alternate Solution:
The complementary equation of $y' - y = x$ is $r - 1 = 0$ which gives $r = 1$. Hence $y_c(x) = c_1e^x$. As well, $y_p(x) = Ax + B$ thus $y_p'(x) = A$. Substituting these into the equation gives $A - (Ax + B) = x$ which gives rise (by comparing coefficients) to the equations $-A = 1$ and $A - B = 0$. Thus $A = -1$ and $B = -1$, i.e. $y_p(x) = -x - 1$. Hence $y(x) = y_c(x) + y_p(x) = c_1e^x - x - 1$. This leads to the same solution as above.

3. (16 points) Solve the initial-value problem

$$y' = x - 2y, \qquad y(1) = 1.$$

Solution:
First, rewrite the equation as $y' + 2y = x$. Then
$I(x) = e^{\int 2dx} = e^{2x}$. Multiply both sides of the equation by this factor to get:

$$e^{2x}y' + 2e^{2x}y = xe^{2x}.$$

This can be written as

$$\frac{d}{dx}(e^{2x}y) = xe^{2x}.$$

Hence

$$\begin{aligned} e^{2x}y &= \int xe^{2x}dx \quad \text{(Let } u = x \text{ and } dv = e^{2x}dx\text{. Then } du = dx \text{ and } v = \frac{1}{2}e^{2x}.\text{)} \\ &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C. \end{aligned}$$

Dividing both sides by e^{2x}:

$$y = \frac{1}{2}x - \frac{1}{4} + Ce^{-2x}.$$

Using the initial value $y(1) = 1$:
$1 = \frac{1}{2} - \frac{1}{4} + Ce^{-2}$ which gives us $C = \frac{e^2}{2}$. Hence

$$y = \frac{1}{2}x - \frac{1}{4} + \frac{e^2}{2}e^{-2x} = \frac{1}{2}x - \frac{1}{4} + \frac{e^{2-2x}}{2}.$$

Alternate Solution:
The complementary equation of $y' + 2y = x$ is $r + 2 = 0$ which gives $r = -2$. Hence $y_c(x) = c_1e^{-2x}$. As well, $y_p(x) = Ax + B$ thus $y_p'(x) = A$. Substituting these into the equation gives $A + 2(Ax + B) = x$ which gives rise (by comparing coefficients) to the equations $2A = 1$ and $A + 2B = 0$. Thus $A = \frac{1}{2}$ and $B = -\frac{1}{4}$, i.e. $y_p(x) = \frac{1}{2}x - \frac{1}{4}$. Hence $y(x) = y_c(x) + y_p(x) = c_1e^{-2x} + \frac{1}{2}x - \frac{1}{4}$. This leads to the same solution as above.

4. (16 points) Solve $y'' - 2y' + y = x^2$.

Solution:
The complementary equation is $r^2 - 2r + 1 = 0$ which factors to $(r-1)^2 = 0$. Hence r has 1 as a double root. Thus $y_c(x) = c_1 e^x + c_2 x e^x$.

Let $y_p(x) = Ax^2 + Bx + C$. Then $y_p'(x) = 2Ax + B$ and $y_p''(x) = 2A$. Substituting these into the original equation:

$$2A - 2(2Ax + B) + Ax^2 + Bx + C = Ax^2 + (B - 4A)x + 2A - 2B + C = x^2.$$

Comparing coefficients of x^2 gives us first that $A = 1$. Comparing the coefficients of x yields the equation $B - 4A = 0$ or $B = 4A$. But $A = 1$, hence $B = 4$. Lastly, comparing the constant terms gives us $2A - 2B + C = 0$ or $C = 2B - 2A$. But $A = 1$ and $B = 4$. Thus $C = 2 \cdot 4 - 2 \cdot 1 = 6$. So $y_p(x) = x^2 + 4x + 6$. Putting the pieces together

$$y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 x e^x + x^2 + 4x + 6.$$

5. (16 points) Solve $y'' - y' - 2y = -2x^2$.

Solution:
The complementary equation is $r^2 - r - 2 = 0$ which factors to $(r-2)(r+1) = 0$. Hence $r = -1, 2$. Thus $y_c(x) = c_1 e^{-x} + c_2 e^{2x}$.

Let $y_p(x) = Ax^2 + Bx + C$. Then $y_p'(x) = 2Ax + B$ and $y_p''(x) = 2A$. Substituting these into the original equation:

$$2A - (2Ax + B) - 2(Ax^2 + Bx + C) = -2Ax^2 + (-2A - 2B)x + 2A - B - 2C = -2x^2.$$

Comparing coefficients of x^2 gives us first that $-2A = -2$ or that $A = 1$. Comparing the coefficients of x yields the equation $-2A - 2B = 0$ or $B = -A$. But $A = 1$, hence $B = -1$. Lastly, comparing the constant terms gives us $2A - B - 2C = 0$ or $C = \frac{1}{2}(2A - B)$. But $A = 1$ and $B = -1$. Thus $C = \frac{1}{2}(2 + 1) = \frac{3}{2}$. So $y_p(x) = x^2 - x + \frac{3}{2}$. Putting the pieces together

$$y(x) = y_c(x) + y_p(x) = c_1 e^{-x} + c_2 e^{2x} + x^2 - x + \frac{3}{2}.$$

6. (16 points) Solve $y'' + 4y = 8x^2$.

Solution:
The complementary equation is $r^2 + 4 = 0$ which means $r = \pm 2i$. Thus $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$.

Let $y_p(x) = Ax^2 + Bx + C$. Then $y_p'(x) = 2Ax + B$ and $y_p''(x) = 2A$. Substituting these into the original equation:

$$2A + 4(Ax^2 + Bx + C) = 4Ax^2 + 4Bx + 2A + 4C = 8x^2.$$

Comparing coefficients of x^2 gives us first that $4A = 8$ or that $A = 2$. Comparing the coefficients of x yields the equation $4B = 0$ or $B = 0$. Lastly, comparing the constant terms gives us $2A + 4C = 0$ or $C = -\frac{A}{2}$. But $A = 2$ thus $C = -1$. So $y_p(x) = 2x^2 - 1$. Putting the pieces together

$$y(x) = y_c(x) + y_p(x) = c_1 \cos 2x + c_2 \sin 2x + 2x^2 - 1.$$

Determine whether the following series converge or diverge. Be sure to state which test(s) you have used. If the series is convergent, find its sum.

7. (11 points)

$$\sum_{n=0}^{\infty} \frac{2}{3^n}.$$

Solution:
This is a geometric series with common ration $r = \frac{1}{3}$. Since $|r| < 1$, the geometric series converges to

$$\sum_{n=0}^{\infty} \frac{2}{3^n} = 2 + \frac{2}{3} + \frac{2}{9} + \cdots = \frac{2}{1 - \frac{1}{3}} = 3.$$

8. (11 points)

$$\sum_{n=0}^{\infty} \frac{3}{2^n}.$$

Solution:
This is a geometric series with common ration $r = \frac{1}{2}$. Since $|r| < 1$, the geometric series converges to

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = 3 + \frac{3}{2} + \frac{3}{4} + \cdots = \frac{3}{1 - \frac{1}{2}} = 6.$$

9. (11 points)

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}.$$

Solution:
This is a *p-series* with $p = \frac{1}{2}$ which is less than or equal to 1, therefore the series diverges.

10. (11 points)

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt[3]{n}}.$$

Solution:
This is a *p-series* with $p = \frac{1}{3}$ which is less than or equal to 1, therefore the series diverges.

11. (11 points)

$$\sum_{n=1}^{\infty} \frac{3}{n^2}.$$

Solution:
This is a *p-series* with $p = 2$ which is greater than 1, therefore the series converges.

12. (11 points)

$$\sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n^2 - n - 2}.$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2 + 2n + 1}{n^2 - n - 2} &= \lim_{n\to\infty} \frac{2n + 2}{2n - 1} \quad \text{(L'Hospital's Rule)} \\
&= \lim_{n\to\infty} \frac{2}{2} \quad \text{(L'Hospital's Rule)} \\
&= 1.
\end{aligned}$$

Since $a_n \not\to 0$ as $n \to \infty$, the series diverges by *the Test for Divergence.*

[NOTE: $n = 2$ gives an "infinite" term, which could be another way to argue that the series diverges.]

13. (20 points) Find a power series representation of the function $f(x) = \ln(1+x^2)$ and determine its interval of convergence.

Solution 1:

$$\begin{aligned}\ln(1-x) &= -\int \frac{1}{1-x}dx \\ &= -\int \left(\sum_{n=0}^{\infty} x^n\right) dx \\ &= C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.\end{aligned}$$

Letting $x = 0$ shows that $C = 0$. Now replace x in the above equation by $-x^2$. Hence

$$\begin{aligned}\ln(1+x^2) &= \ln(1-[-x^2]) \\ &= -\sum_{n=0}^{\infty} \frac{[-x^2]^{n+1}}{n+1} \\ &= -\sum_{n=0}^{\infty} \frac{[(-1)(x^2)]^{n+1}}{n+1} \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x^2)^{n+1}}{n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+2}x^{2n+2}}{n+1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{n}.\end{aligned}$$

Solution 2:

$$\begin{aligned}f'(x) &= \frac{2x}{1+x^2} \\ &= 2x \cdot \frac{1}{1-[-x^2]} \\ &= 2x\sum_{n=0}^{\infty}[-x^2]^n \\ &= 2x\sum_{n=0}^{\infty}(-1)^n x^{2n} \\ &= 2\sum_{n=0}^{\infty}(-1)^n x^{2n+1}\end{aligned}$$

Therefore

$$\begin{aligned} f(x) &= \int \left(2\sum_{n=0}^{\infty}(-1)^n x^{2n+1} \right) dx \\ &= 2\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+2} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n}. \end{aligned}$$

In each method we used the geometric series replacing x by $-x^2$. Thus we have convergence when $|-x^2| = |x|^2 < 1$, i.e. $R = 1$. Checking the endpoints, let $x = -1$. Then the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{n}.$$

Note that $b_{n+1} \leq b_n$ for all $n \geq 1$ and that $b_n \to 0$. Hence the series converges by the *Alternating Series Test.* Now let $x = 1$. Then the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Note that $b_{n+1} \leq b_n$ for all $n \geq 1$ and that $b_n \to 0$. Hence the series converges by the *Alternating Series Test.* Therefore

$$I = [-1, 1].$$

14. (20 points) Find a power series representation of the function $f(x) = x^2 \cdot \ln(1 + x^2)$ and determine its interval of convergence.
Solution:
Same as previous question except that we must multiply the power series by x^2.

15. (20 points) Find a power series representation of the function $f(x) = \frac{x^3}{(1+x^2)^2}$ and determine its interval of convergence.

Solution 1:

$$(1-x)^{-1} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{so}$$
$$(1-x)^{-2} = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}.$$

So

$$\begin{aligned}\frac{x^3}{(1+x^2)^2} &= x^3 \cdot \frac{1}{(1-[-x^2])^2} \\ &= x^3 \sum_{n=1}^{\infty} n[-x^2]^{n-1} \\ &= x^3 \sum_{n=1}^{\infty} n(-1)^{n-1} x^{2n-2} \\ &= \sum_{n=1}^{\infty} n(-1)^{n-1} x^{2n+1}.\end{aligned}$$

Solution 2:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$
$$\text{differentiating both sides: } \frac{-2x}{(1+x^2)^2} = \sum_{n=0}^{\infty} (-1)^n (2n) x^{2n-1}.$$

So

$$\begin{aligned}\frac{x^3}{(1+x^2)^2} &= \frac{x^2}{-2} \cdot \frac{-2x}{(1+x^2)^2} \\ &= \frac{x^2}{-2} \sum_{n=1}^{\infty} (-1)^n (2n) x^{2n-1} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} n x^{2n+1}.\end{aligned}$$

Each method used the geometric series replacing x by $-x^2$. Thus we have convergence when $|-x^2| = |x|^2 < 1$, i.e. $R = 1$. Checking the endpoints, let $x = -1$. Then the series is

$$\sum_{n=1}^{\infty} n(-1)^{n-1}(-1)^{2n+1} = \sum_{n=1}^{\infty}(-1)^{3n}n$$

Unfortunately we must check this by looking at the partial sums (the lack of absolute convergence tells us nothing!). So

$$\begin{aligned} S_{2k} &= (-1+2) + (-3+4) + \cdots + (-[2k-1] + 2k) \\ &= 1 + 1 + \cdots + 1 \\ &= k. \end{aligned}$$

Thus the limit of the (even) partial sums diverges, so the series diverges.

OR

$\lim_{n\to\infty}(-1)^{3n}n$ D.N.E., therefore, by The Test for Divergence, the series $\sum_{n=1}^{\infty}(-1)^{3n}n$ diverges.

Regarding $x = 1$, the series is

$$\sum_{n=1}^{\infty} n(-1)^{n-1}1^{2n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}n.$$

A similar argument as above shows this series also diverges. Hence

$$I = (-1, 1).$$

16. (15 points) Find the Taylor series for $f(x) = \ln x$ centered at $a = 1$ (assume that $f(x)$ has a power series expansion; i.e. you do not have to show that $R_n(x) \to 0$).

Solution:

$$\begin{aligned} f(x) &= \ln x & &\text{so } f(1) = 0 \\ f'(x) &= x^{-1} & &\text{so } f'(1) = 1 \\ f''(x) &= -x^{-2} & &\text{so } f''(1) = -1 \\ f'''(x) &= 2x^{-3} & &\text{so } f'''(1) = 2 \\ f^{(4)}(x) &= -3 \cdot 2x^{-4} & &\text{so } f^{(4)}(1) = -6 \\ &\vdots & &\vdots \end{aligned}$$

So

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= 0 + \frac{1}{1}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4 + \cdots \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(n-1)!}{n!}(x-1)^n \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}. \end{aligned}$$

17. (15 points) Find the Taylor series for $f(x) = \sin x$ centered at $a = \pi$ (assume that $f(x)$ has a power series expansion; i.e. you do not have to show that $R_n(x) \to 0$).
Solution:

$$\begin{aligned} f(x) &= \sin x & \text{so } f(\pi) &= 0 \\ f'(x) &= \cos x & \text{so } f'(\pi) &= -1 \\ f''(x) &= -\sin x & \text{so } f''(\pi) &= 0 \\ f'''(x) &= -\cos x & \text{so } f'''(\pi) &= 1 \\ f^{(4)}(x) &= \sin x & \text{so } f^{(4)}(\pi) &= 0 \\ &\vdots & &\vdots \end{aligned}$$

So

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n \\ &= 0 + -1(x-\pi) + \frac{0}{2!}(x-\pi)^2 + \frac{1}{3!}(x-\pi)^3 + \frac{0}{4!}(x-\pi)^4 + \cdots \\ &= -1(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \cdots \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{(x-\pi)^{2n-1}}{(2n-1)!}. \end{aligned}$$

18. (15 points) Find the Taylor series for $f(x) = \cos x$ centered at $a = \pi$ (assume that $f(x)$ has a power series expansion; i.e. you do not have to show that $R_n(x) \to 0$).
Solution:

$$\begin{aligned} f(x) &= \cos x & &\text{so } f(\pi) = -1 \\ f'(x) &= -\sin x & &\text{so } f'(\pi) = 0 \\ f''(x) &= -\cos x & &\text{so } f''(\pi) = 1 \\ f'''(x) &= \sin x & &\text{so } f'''(\pi) = 0 \\ f^{(4)}(x) &= \cos x & &\text{so } f^{(4)}(\pi) = -1 \\ &\vdots & &\vdots \end{aligned}$$

So

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n \\ &= -1 + 0(x-\pi) + \frac{1}{2!}(x-\pi)^2 + \frac{0}{3!}(x-\pi)^3 + \frac{-1}{4!}(x-\pi)^4 + \cdots \\ &= -1 + \frac{1}{2!}(x-\pi)^2 - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n}}{(2n)!}. \end{aligned}$$