KEY **Math 115-009 Calculus 1**

CIRCLE ONE: Grade my WORK / ANSWERS **Exam 2**

Answer each of the following. If you choose to have your work graded, please show all work (no work=no credit).

Find the following derivatives:

1. (10 points each) Differentiate each of the following:

   (a) $y = \frac{e^x}{1+x^2}$
   Solution:

$$\begin{aligned} y' &= \frac{(1+x^2)(e^x) - (e^x)(2x)}{(1+x^2)^2} \\ &= \frac{(e^x)(x^2-2x+1)}{(1+x^2)^2}. \end{aligned}$$

   (b) $y = x^{\cos 3x}$
   Solution:
   Taking the natural log of both sides gives us

$$\ln y = \ln x^{\cos 3x} = \cos 3x \cdot \ln x.$$

   Differentiating

$$\begin{aligned} \frac{1}{y}y' &= \cos 3x \cdot \frac{1}{x} + \ln x \cdot (-\sin 3x) \cdot 3 \\ &= \frac{\cos 3x}{x} - 3\ln x \cdot \sin 3x. \end{aligned}$$

   Hence

$$y' = y\left(\frac{\cos 3x}{x} - 3\ln x \cdot \sin 3x\right) = x^{\cos 3x}\left(\frac{\cos 3x}{x} - 3\ln x \cdot \sin 3x\right).$$

   (c) $y = \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}$
   Solution:
   Taking the natural log of both sides gives us

$$\begin{aligned} \ln y &= \ln \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} \\ &= \ln(x^2+1)^4 - \left[\ln(2x+1)^3 + \ln(3x-1)^5\right] \\ &= 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1). \end{aligned}$$

   Differentiating

$$\begin{aligned} \frac{1}{y}y' &= 4 \cdot \frac{1}{x^2+1} \cdot 2x - 3 \cdot \frac{1}{2x+1} \cdot 2 - 5 \cdot \frac{1}{3x-1} \cdot 3 \\ &= \frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}. \end{aligned}$$

Hence

$$
\begin{aligned}
y' &= y\left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right)\\
&= \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right)\\
&= \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\left(\frac{8x[2x+1][3x-1] - 6[x^2+1][3x-1] - 15[x^2+1][2x+1]}{(x^2+1)(2x+1)(3x-1)}\right)\\
&= \frac{(x^2+1)^3}{(2x+1)^4(3x-1)^6}\left(8x[6x^2+x-1] - 6[3x^3-x^2+3x-1] - 15[2x^3+x^2+2x+1]\right)\\
&= \frac{(x^2+1)^3}{(2x+1)^4(3x-1)^6}\left(48x^3+8x^2-8x-18x^3+6x^2-18x+6-30x^3-15x^2-30x-15\right)\\
&= \frac{(x^2+1)^3}{(2x+1)^4(3x-1)^6}\left(-x^2-56x-9\right).
\end{aligned}
$$

2. (12 points) Find an equation of the line tangent to $x^2+4xy+y^2=13$ at the point $(2,1)$.
Solution:
Differentiating implicitly gives us

$$2x + 4x\cdot\frac{dy}{dx} + 4y + 2y\cdot\frac{dy}{dx} = 0.$$

A little algebra gives

$$\frac{dy}{dx}(4x+2y) = -2x-4y.$$

Hence

$$\frac{dy}{dx} = \frac{-2x-4y}{4x+2y} = \frac{-x-2y}{2x+y}.$$

Thus the slope of the tangent line at the point $(2,1)$ is $m = \frac{-2-2}{4+1} = -\frac{4}{5}$. So an equation of the tangent line is

$$y = 1 + -\frac{4}{5}(x-2) = -\frac{4}{5}x + \frac{13}{5}.$$

3. The cost, in dollars, of producing ax units of a certain commodity is

$$C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3.$$

(a) (6 points) Find the marginal cost function $C'(x)$.
Solution:

$$C'(x) = 2 - 0.04x + 0.00021x^2.$$

(b) (3 points) Find $C'(100)$ and explain its meaning.
Solution:

$$C'(100) = 2 - 0.04(100) + 0.00021(100)^2 = 2 - 4 + 2.1 = 0.1.$$

This means that it will cost an additional \$0.10 to produce the 101^{st} item. Note that $C(101) - C(100) = 0.10107$.

4. A particle moves on a vertical line so that its coordinate at time t is

$$y = t^3 - 12t + 3$$

where $t \geq 0$.

(a) (6 points) Find the velocity function.

$$v(t) = y' = 3t^2 - 12.$$

(b) (4 points) When is the particle moving upward and when is it moving downward?
Note that $v(t) = 3(t^2 - 4) = 3(t-2)(t+2)$. Now $t - 2 < 0$ when $t < 2$ and $t - 2 > 0$ when $t > 2$. As well, $t + 2 < 0$ when $t < -2$ and $t + 2 > 0$ when $t > -2$. Hence

v(t)	+	-	+
t-2	-	-	+
t+2	-	+	+
interval:	$(-\infty, -2)$	(-2,2)	$(2,\infty)$

Thus the particle is moving downward when $0 < t < 2$ and upward when $t > 2$.

(c) (6 points) Find the acceleration function.

$$a(t) = v'(t) = 6t.$$

(d) (2 points) When is the particle speeding up?
$a(t) > 0$ when $t > 0$. Thus the particle is speeding up when $t > 0$.

(e) (2 points) When is the particle slowing down?
Since $a(t)$ is never negative for $t > 0$, the particle is never slowing down.

5. (14 points) A paper cup has the shape of a cone with height $10cm$ and radius $4cm$ (at the top). If water is poured into the cup at a rate of $2cm^3/s$, how fast is the water level rising when the water is $4cm$ deep? (The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)
Solution:
Let $h =$ the height of the water in the cone and let $r =$ the radius of the circle at the top of the water in the cone. Then $\frac{10}{4} = \frac{h}{r}$ (Draw a picture). This gives us $r = \frac{2h}{5}$. Hence

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h = \frac{4\pi h^3}{75}.$$

Thus

$$\frac{dV}{dt} = \frac{4\pi h^2}{25}\frac{dh}{dt}.$$

We have that $\frac{dV}{dt} = 2$, so

$$\frac{dh}{dt} = \frac{25\frac{dV}{dt}}{4\pi h^2} = \frac{25 \cdot 2}{4\pi \cdot 4^2} = \frac{25}{32\pi}.$$

6. (a) (10 points) Find the linearization of $f(x) = \sqrt[4]{1+8x}$ near $x = 0$.
Solution:
$f(x) = (1+8x)^{\frac{1}{4}}$, so $f'(x) = \frac{1}{4}\cdot(1+8x)^{-\frac{3}{4}}\cdot 8 = 2(1+8x)^{-\frac{3}{4}}$. Thus the slope of the line tangent to $f(x)$ at $x = 0$ is $m = f'(0) = \frac{2}{1} = 2$. So the equation of the tangent line is $y = f(0) + 2(x - 0) = 1 + 2x$. Thus the linearization $L(x)$ is $\sqrt[4]{1+8x} \approx 2x + 1$ near $x = 0$.

(b) (5 points) Use this linearization to approximate $\sqrt[4]{1.08}$.
Solution:
Notice that $\sqrt[4]{1.08} = f(0.01) \approx L(0.01) = 2(0.01) + 1 = 1.02$.
Note that $\sqrt[4]{1.08} \approx 1.01942654691$.