1. Recurrence Relations

Definition 1.1.
*A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence. A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.*

Example 1.2.
Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if

(1) $a_n = 0$
Solution: $0 = -3 \cdot 0 + 4 \cdot 0$ ✓

(2) $a_n = 1$
Solution: $1 = -3 \cdot 1 + 4 \cdot 1$ ✓

(3) $a_n = (-4)^n$
Solution:

$$\begin{aligned} -3 \cdot (-4)^{n-1} + 4 \cdot (-4)^{n-2} &= (-4)^{n-2}[-3 \cdot -4 + 4] && (1) \\ &= (-4)^{n-2} \cdot 4^2 && (2) \\ &= (-4)^n && (3) \\ &= a_n \quad ✓ \end{aligned}$$

(4) $a_n = 2 \cdot (-4)^n + 3$
Solution:

$$\begin{aligned} -3[2(-4)^{n-1} + 3] + 4[2(-4)^{n-2} + 3] &= -6(-4)^{n-1} - 9 + 8(-4)^{n-2} + 12 && (4) \\ &= (-4)^{n-2}[-6(-4) + 8] + 3 && (5) \\ &= (-4)^{n-2}[2(-4)^2] + 3 && (6) \\ &= 2(-4)^n + 3 && (7) \\ &= a_n \quad ✓ \end{aligned}$$

Example 1.3 (The Tower of Hanoi)**.**
A puzzle consists of three pages mounted to a board with a stack of disks of different sizes. Initially the disks are placed on the first peg in order of size, with the largest on the bottom. Disks may be moved to any peg with one condition: no disk may be placed on top of a disk of

smaller size. What is the smallest number of legal moves involved in transferring the stack from the first peg to a different (say third) peg?

Solution:
Let $H_n =$ the number of moves needed to solve the puzzle with n disks. Then $H_n = 2H_{n-1} + 1$ with the initial condition $H_n = 1$. Hence we get the following iterated solution:

$$
\begin{aligned}
H_n &= 2H_{n-1} + 1 && (8)\\
&= 2(2H_{n-2} + 1) + 1 && (9)\\
&= 2^2 H_{n-2} + 2 + 1 && (10)\\
&= 2^2(2H_{n-3} + 1) + 2 + 1 && (11)\\
&= 2^3 H_{n-3} + 2^2 + 2 + 1 && (12)\\
&\vdots\\
&= 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1 && (13)\\
&= 2^n - 1. && (14)
\end{aligned}
$$

Example 1.4.
Find the solution to $a_n = a_{n-1} + n$ with $a_0 = 0$ using an iterative approach as in the Tower of Hanoi example.

Solution:

$$
\begin{aligned}
a_n &= a_{n-1} + n && (15)\\
&= [a_{n-2} + n - 1] + n && (16)\\
&= [a_{n-3} + n - 2] + (n-1) + n && (17)\\
&\vdots\\
&= a_0 + \cdots + (n-2) + (n-1) + n && (18)\\
&= 1 + \cdots + (n-2) + (n-1) + n && (19)\\
&= \frac{n(n+1)}{2}. && (20)
\end{aligned}
$$

2. Solving Recurrence Relations

Definition 2.1.
A recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$

*where each $c_i \in \mathbb{R}$ is a **linear homogenous recurrence relation of degree k with constant coefficients**.*

Suppose $a_n = r^n$ is a solution to a linear homogenous recurrence relation. Note that $a_n = r^n$ is a solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \tag{21}$$

if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}. \tag{22}$$

A little algebra yields

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} + \cdots + c_{k-1} r - c_k. \tag{23}$$

Hence $a_n = r^n$ is a solution of (21) if and only if r is a solution of (23).

Because of its importance, the equation (23) is called the **characteristic equation** and its solutions are called the **characteristic roots**.

Theorem 2.2.
Let c_1 and c_2 be real numbers and suppose the equation $r^2 - c_1 r - c_2 = 0$ has two distinct roots. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$ for $n = 0, 1, \ldots$ where α_1 and α_2 are constants.

Proof.
($\Leftarrow$) Suppose

$$a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n. \tag{24}$$

Since $r^2 - c_1 r - c_2 = 0$ we have $r^2 = c_1 r + c_2$. r_1 and r_2 are roots of this equation hence

$$\begin{aligned} {r_1}^2 &= c_1 r_1 + c_2 \text{ and} \quad (25) \\ {r_2}^2 &= c_1 r_2 + c_2. \quad (26) \end{aligned}$$

These equations give us

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 {r_1}^{n-1} + \alpha_2 {r_2}^{n-1}) + c_2(\alpha_1 {r_1}^{n-2} + \alpha_2 {r_2}^{n-2}) \quad (27) \\ &= c_1 \alpha_1 {r_1}^{n-1} + c_2 \alpha_1 {r_1}^{n-2} + c_1 \alpha_2 {r_2}^{n-1} + c_2 \alpha_2 {r_2}^{n-2}) \quad (28) \\ &= \alpha_1 {r_1}^{n-2}(c_1 r_1 + c_2) + \alpha_2 {r_2}^{n-2}(c_1 r_2 + c_2) \quad (29) \\ &= \alpha_1 {r_1}^{n-2} {r_1}^2 + \alpha_2 {r_2}^{n-2} {r_2}^2 \quad (30) \\ &= \alpha_1 {r_1}^n + \alpha_2 {r_2}^n \quad (31) \\ &= a_n \quad (32) \end{aligned}$$

Hence we have shown that the sequence $\{a_n\}$ with $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$ is a solution of the recurrence relation.

($\Rightarrow$) We now want to show that every sequence a_n which is a solution of the recurrence relation is of the form $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$. Suppose that $a_0 = C_0$ and $a_1 = C_1$. We then will need

$$\begin{aligned} a_0 = C_0 &= \alpha_1 + \alpha_2 \qquad \text{and} \quad (33) \\ a_1 = C_1 &= \alpha_1 r_1 + \alpha_2 r_2. \quad (34) \end{aligned}$$

Solving (33) gives us

$$\alpha_2 = C_0 - \alpha_1.$$

Substituting into (34) we get

$$C_1 = \alpha_1 r_1 + (C_0 - \alpha_1) r_2 = \alpha_1(r_1 - r_2) + C_0 r_2.$$

In other words,

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}.$$

Thus

$$\begin{aligned}\alpha_2 &= C_0 - \alpha_1 \\ &= C_0 \cdot \frac{r_1 - r_2}{r_1 - r_2} - \frac{C_1 - C_0 r_2}{r_1 - r_2} \\ &= \frac{C_0 r_1 - C_0 r_2 - C_1 + C_0 r_2}{r_1 - r_2} \\ &= \frac{C_0 r_1 - C_1}{r_1 - r_2}\end{aligned}$$

Note that for each of these values it is necessary that $r_1 \neq r_2$, i.e. the characteristic equation has distinct roots.

We now have values for α_1 and α_2 which satisfy the initial conditions and meet the recurrence relation and, thus, uniquely determine the sequence. Hence it follows that $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$.

□

Note that the theorem we just proved is valid if c_1 and c_2 are complex numbers.

Example 2.3.
Solve the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$ subject to the initial conditions $a_0 = 3$ and $a_1 = 4$.
First step: the ***characteristic equation****.*
We have

$$\begin{aligned} r^n = r^{n-1} + 6r^{n-2} &\Leftrightarrow r^2 = r + 6 && (35)\\ &\Leftrightarrow r^2 - r - 6 = 0 && (36)\\ &\Leftrightarrow (r-3)(r+2) = 0 && (37)\end{aligned}$$

Thus $r_1 = 3$ and $r_2 = -2$.

Second step: obtaining the constants.
Using $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$, we get the following system of equations:

$$\begin{aligned} 3 &= \alpha_1 3^0 + \alpha_2(-2)^0 = \alpha_1 + \alpha_2 && (38)\\ 4 &= \alpha_1 3^1 + \alpha_2(-2)^1 = 3\alpha_1 - 2\alpha_2 && (39)\end{aligned}$$

Multiplying the top equation by 2:

$$\begin{aligned} 6 &= 2\alpha_1 + 2\alpha_2 && (40)\\ 4 &= 3\alpha_1 - 2\alpha_2 && (41)\end{aligned}$$

Adding these gives

$$10 = 5\alpha_1$$

so $\alpha_1 = 2$ *and thus* $\alpha_2 = 1$.

Hence the general solution is

$$a_n = 2 \cdot 3^n + (-2)^n.$$

As a check, note that

$$a_0 = 2 \cdot 3^0 + (-2)^0 = 2 + 1 = 3 \text{ and} \tag{42}$$

$$a_1 = 2 \cdot 3^1 + (-2)^1 = 6 - 2 = 4 \tag{43}$$

Theorem 2.4.
Let c_1, $c_2 \in \mathbb{R}$ *with* $c_2 \neq 0$. *Suppose the characteristic equation*

$$r^2 - c_1 r - c_2 = 0$$

has a single root r_0 *of multiplicity* 2. *Then a sequence* $\{a_n\}$ *is s solution of the recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \alpha_1 {r_0}^n + \alpha_2 n {r_0}^n,$$

for $n = 0, 1, 2, \ldots$ *where* α_1 *and* $alpha_2$ *are constants.*

Example 2.5.
Solve the recurrence relation given by $a_n = 4a_{n-1} - 4a_{n-2}$ *with* $a_0 = 5$ *and* $a_1 = 24$.

Solution:
The characteristic equation is: $r^2 = 4r - 4$ which is equivalent to $r^2 - 4r - 4 = 0$ which has a single root $r = 2$ of multiplicity 2. Hence the solutions are given by

$$5 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 0 \cdot 2^0 \text{ and} \tag{44}$$

$$24 = \alpha_1 \cdot 2^1 + \alpha_2 \cdot 1 \cdot 2^1. \tag{45}$$

Thus $\alpha_1 = 5$ and $\alpha_2 = 7$. Hence the general solution is

$$a_n = 5 \cdot 2^n + 7n \cdot 2^n.$$

Theorem 2.6.
Let c_1, c_2, $\ldots$, $c_k \in \mathbb{R}$. *Suppose the characteristic equation*

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 {r_1}^n + \cdots + \alpha_k {r_k}^n$$

for $n = 0, 1, 2, \ldots,$ where $\alpha_1, \ldots, \alpha_k$ are constants.

Since we will be solving higher degree polynomials, the following is a helpful reminder from Precalculus:

Theorem 2.7 (The Rational Roots Test)**.**
Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial with real coefficients and $a_n \neq 0$. If $P(x)$ has rational coefficients they are of the form $\pm\frac{p}{q}$ where $p|a_0$ and $q|a_n$.

Example 2.8.
Solve the recurrence $a_n = 2a_{n-1} + 5a_{n-2} + 6a_{n-3}$ with the initial conditions given by $a_0 = 3$, $a_1 = 2$, and $a_2 = 14$.