Linear Programming
Math 70 – 460 (Mathematical Models for Consultants)
Carnegie Mellon University - Spring 2009

Please submit the following problems at the beginning of class Thursday, April 9.

(1) A delivery company has four factories (A,B,C,D) in different locations and four warehouses (W,X,Y,Z) also in different locations. The cost to deliver a unit of its product from a particular factory to a particular warehouse is given by the following table:

Table 1. Shipping Costs

	W	X	Y	Z
A	10	30	25	15
B	20	15	20	10
C	10	30	20	20
D	30	40	35	45

Production levels at the factories are:

Table 2. Factory Production

Factory	Supply
A	1500
B	600
C	1400
D	1100

Demand levels for the warehouse are:

Table 3. Warehouse Demand

Warehouse	Demand
W	1000
X	1200
Y	1500
Z	900

Use Excel to determine what shipping schedule should be followed in order to meet each warehouse's demand while minimizing shipping costs.

The Minimum cost is \$ 94000 which is reached when $AW = 600$, $AZ = 900$, $BX = 600$, $CW = 400$, $CY = 1000$, $DX = 600$, and $DY = 500$. All the rest of the variables are 0 (see attached file).

(2) A company makes dumpsters and has a customer that would like a rectangular dumpster that has a volume of $48ft^3$. Suppose the following costs are associated with constructing the dumpster:
   (a) The sides, front and back are to be made from 12-gauge steel sheets that cost \$0.70 per square foot (this cost includes all labor to cut and bend the steel).
   (b) The bottom is to be made from 10-gauge steel sheets that cost \$0.90 per square foot.
   (c) Welding costs \$0.18 per foot for labor and materials combined.
   (d) The cost of a lid for each dumpster is independent of the size of the dumpster (thus we will ignore this cost).

   What dimensions should we make the dumpster in order to meet our customer's demands yet minimize costs?

   Please do this problem using
   (a) partial derivatives (include using the Second Derivative Test in order to justify that we do have a minimum), and
   (b) Lagrange multipliers (you may use a calculator, software, etc. to solve the system of equations).

Solution:
Let x be the length, y the width, and z the height. Our job then is to

$$\begin{aligned} \text{Minimize } C(x,y,z) =& 1.4xz + 1.4yz + .9xy + .36x + .36y + .72z \\ \text{Subject to } V(x,y,z) =& xyz = 48, \\ & x,y,z > 0. \end{aligned}$$

Method 1: Using Multivariable Calculus
We first solve the objective function for z, obtaining $z = \frac{48}{xy}$. We now substitute this into the objective function:

$$\begin{aligned} C(x,y) &= 1.4x\frac{48}{xy} + 1.4y\frac{48}{xy} + .9xy + .36x + .36y + .72\frac{48}{xy} \\ &= 67.2y^{-1} + 67.2x^{-1} + .9xy + .36x + .36y + 34.56x^{-1}y^{-1} \end{aligned}$$

Hence

$$\frac{\partial C}{\partial x} = C_x = -67.2x^{-2} + .9y + .36 - 34.56x^{-2}y^{-1} \text{ and} \tag{1}$$

$$\frac{\partial C}{\partial y} = C_y = -67.2y^{-2} + .9x + .36 - 34.56x^{-1}y^{-2}. \tag{2}$$

The partial derivatives are undefined for $x = 0 = y$, but these values are not feasible.

In setting each of equations (1) and (2) equal to 0, the careful observer will note that these equations are symmetric in x and y and this implies that $x = y$.

Substituting x in for y in equation (1) and setting it equal to 0 gives

$$-67.2x^{-2} + .9x + .36 - 34.56x^{-3} = 0. \tag{3}$$

Multiplying by x^3 and rearranging

$$.9x^4 + .36x^3 - 67.2x - 34.56 = 0. \tag{4}$$

Using Solver or a graphing calculator yields that $x = 4.24$. Hence $y = 4.24$ and $z = \frac{48}{(4.24)(4.24)} = 2.67$.

Method 2: Using Lagrange Multipliers
Taking partial derivatives we have

$$C_x = 1.4z + 0.9y + 0.36 \tag{5}$$
$$C_y = 1.4z + 0.9x + 0.36 \tag{6}$$
$$C_z = 1.4x + 1.4y + 0.72 \tag{7}$$
$$V_x = yz \tag{8}$$
$$V_y = xz \tag{9}$$
$$V_z = xy \tag{10}$$

So the system of equations that we must solve is

$$1.4z + 0.9y + 0.36 = \lambda yz \tag{11}$$
$$1.4z + 0.9x + 0.36 = \lambda xz \tag{12}$$
$$1.4x + 1.4y + 0.72 = \lambda xy \tag{13}$$
$$xyz = 48 \tag{14}$$

You may use Solver or any other device to find the solutions. If you want to work out the details, equations (11) and (12) again imply (by symmetry) that $x = y$. making this substitution reduces the above system to

$$1.4z + 0.9x + 0.36 = \lambda xz \tag{15}$$

$$2.8x + 0.72 = \lambda x^2 \tag{16}$$

$$x^2 z = 48 \tag{17}$$

Equation (16) implies that $\lambda = \frac{2.8}{x} + \frac{0.72}{x^2}$ and solving equation (17) yields that $z = \frac{48}{x^2}$. Substituting these into (15) and doing some simplifying yields that

$$.9x^4 + .36x^3 - 67.2x - 34.56 = 0 \tag{18}$$

which exactly the same equation we had in method 1.