Homework 3 Solutions
Math 70 – 460 (Mathematical Models for Consultants)
Carnegie Mellon University - Spring 2010

Please submit the following problems at the beginning of class Tuesday, April 20.

(1) Use the ***Arithmetic-Geometric Mean*** inequality (AGM) to solve the following optimization problems [30 points each]:
(a)

$$\begin{aligned} \text{Minimize: } & P(x,y,z) = x^2 + y + z \\ \text{Subject to: } & xyz = 1 \\ & x, y, z > 0 \end{aligned}$$

Solution:

$$\begin{aligned} x^2 + y + z &= x^2 + \frac{y}{2} + \frac{y}{2} + \frac{z}{2} + \frac{z}{2} && (1) \\ &= 5\left(\frac{x^2}{5} + \frac{y}{10} + \frac{y}{10} + \frac{z}{10} + \frac{z}{10}\right) && (2) \\ \text{(by the AGM)} \quad &\geq 5\left(x^2 \cdot \frac{y}{2} \cdot \frac{y}{2} \cdot \frac{z}{2} \cdot \frac{z}{2}\right)^{\frac{1}{5}} && (3) \\ &= 5 \cdot 2^{-\frac{4}{5}}(xyz)^{\frac{2}{5}} && (4) \\ &= \frac{5}{\sqrt[5]{16}} \qquad \text{(since } xyz = 1\text{)} && (5) \\ &\approx 2.8717 \end{aligned}$$

This maximum occurs when

$$x^2 = \frac{y}{2} = \frac{z}{2},$$

i.e. when $y = z$ and $y = 2x^2$. Substituting

$$\begin{aligned} x(2x^2)(2x^2) &= 1 && (6) \\ 4x^5 &= 1 && (7) \end{aligned}$$

So

$$x = \sqrt[5]{\frac{1}{4}} = 2^{-\frac{2}{5}} \approx 0.7578$$

and

$$y = z = 2x^2 = 2^{\frac{1}{5}} \approx 1.1486.$$

(b)

$$\begin{aligned} \text{Maximize: } & Q(x,y,z) = xyz \\ \text{Subject to: } & 3x + 4y + 12z = 1 \\ & x, y, z > 0 \end{aligned}$$

Solution:

$$\begin{aligned} 1 &= 3x + 4y + 12z && (8) \\ &= 3\left(\frac{3x}{3} + \frac{4y}{3} + \frac{12z}{3}\right) && (9) \\ \text{by the AGM } &\geq 3\left(3x \cdot 4y \cdot 12z\right)^{\frac{1}{3}} && (10) \\ &= 3(3^2 \cdot 2^4 \cdot xyz)^{\frac{1}{3}} && (11) \\ &= 3^{\frac{5}{3}} \cdot 2^{\frac{4}{3}} \cdot (xyz)^{\frac{1}{3}} && (12) \end{aligned}$$

therefore

$$\begin{aligned} xyz &\leq (3^{-\frac{5}{3}} 2^{-\frac{4}{3}})^3 && (13) \\ &= 3^{-5} \cdot 2^{-4} && (14) \end{aligned}$$

So max Q is $3^{-5} \cdot 2^{-4} \approx 0.0002572$. This maximum occurs when

$$3x = 4y = 12z,$$

i.e.

$$y = 3z, x = 4z.$$

Substituting

$$3(4z) + 4(3z) + 12z = 1$$

hence

$$36z = 1$$

and we have

$$\begin{aligned} z &= \frac{1}{36} \\ y &= \frac{1}{12} \\ x &= \frac{1}{9}. \end{aligned}$$

(2) Use Excel's Solver to answer the above questions [15 points each]. Note:
-since there is no ">" in the constraints, use $x > 0.00000001$ or something similar

-you must have initial values for x, y, z, so for (a) initially try $x = y = z = 1$. Then try $x = 100, y = z = 0.1$ and watch what happens.

(3) **Extra Credit** [20 points for (a), 10 points for (b)]

(a) By hand, solve

$$\text{Minimize: } g(x,y) = \frac{1000}{xy} + 2x + 2y + xy$$
$$\text{Subject to: } x, y > 0.$$

The feasible set in this situation consists of more than 1 vector. You may use a calculator or computer software to solve any system of equations you encounter.

(b) Use Solver to answer part (a).

Solution: The dual is to maximize

$$v(\delta_1, \delta_2, \delta_3, \delta_4) = \left(\frac{1000}{\delta_1}\right)^{\delta_1} \left(\frac{2}{\delta_2}\right)^{\delta_2} \left(\frac{2}{\delta_3}\right)^{\delta_3} \left(\frac{1}{\delta_4}\right)^{\delta_4}$$

Subject to:

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 = 1 \text{ (the Normality Constraint)} \quad (15)$$
$$-\delta_1 + \delta_2 + \delta_4 = 0 \text{ (Orthogonality Constraint)} \quad (16)$$
$$-\delta_1 + \delta_3 + \delta_4 = 0 \text{ (Orthogonality Constraint)} \quad (17)$$

and $\delta_1, \delta_2, \delta_3, \delta_4 > 0$ (Positivity Constraint).

The middle two equations imply that $\delta_2 = \delta_3$, so

$$\delta_2 = \delta_3 = \delta_1 - \delta_4.$$

Put $\delta_4 = t$. Then substituting into the Normality Constraint give us:

$$\delta_1 + (\delta_1 - \delta_4) + (\delta_1 - \delta_4) + \delta_4 = 1$$

So

$$3\delta_1 = 1 + \delta_4 = 1 + t$$

therefore

$$\delta_1 = \frac{1+t}{3}$$
$$\delta_2 = \delta_3 = \frac{1+t}{3} - t = \frac{1+t-3t}{3} = \frac{1-2t}{3}$$
$$\text{and } \delta_4 = t.$$

So we must maximize

$$v(\delta_1, \delta_2, \delta_3, \delta_4) = \left(\frac{3000}{1+t}\right)^{\frac{1+t}{3}} \left(\frac{6}{1-2t}\right)^{\frac{2(1-2t)}{3}} \left(\frac{1}{t}\right)^{t}.$$

Since the natural logarithm function is a continuous increasing function, this maximum will occur at the same values that maximize

$$\frac{1+t}{3}\ln\left(\frac{3000}{1+t}\right) + \frac{2(1-2t)}{3}\ln\left(\frac{6}{1-2t}\right) + t\ln\left(\frac{1}{t}\right)$$

which we can solve (unpleasantly) by using calculus or a computer.

Using Solver gives the minimum of $g(x, y)$ as 84.8207 which occurs when $x = y = 5.1828$.

(4) A location problem: Rappaport Communications (20 points)

- Rappaport Communications provides cellular phone service in several midwestern states.
- They seek to expand their operations by providing intercity service between four cities in northern Ohio.
- To do this, a new tower must be built to accommodate the existing towers in the cities.
- The tower will have a 40 mile transmission radius.
- Rappaport would like to minimize the distance between the new tower and the existing towers.

We can express the location of each tower as an ordered pair (x, y) where x represents the distance east (in miles) the tower is from an arbitrary, but fixed reference point, and y represents the distance north (in miles) the tower is from the same reference point.

The Cleveland tower is located at $(5, 45)$, the Akron tower at $(12, 21)$, the Canton tower at $(17, 5)$, and the Youngstown tower at $(52, 21)$. Let (x, y) represent the location of the new tower.

Use Solver to locate the optimal position (the one that minimizes the total distance) of the new tower. Recall that the

distance between points (a, b) and (x, y) is given by
$d = \sqrt{(x-a)^2 + (y-b)^2}$.

<u>*Solution:*</u>
Minimize
$d = \sqrt{(x-5)^2 + (y-45)^2} + \sqrt{(x-12)^2 + (y-21)^2} + \sqrt{(x-17)^2 + (y-5)^2} + + \sqrt{(x-52)^2 + (y-21)^2}$ Subject to

$$\sqrt{(x-5)^2 + (y-45)^2} \leq 40$$
$$\sqrt{(x-12)^2 + (y-21)^2} \leq 40$$
$$\sqrt{(x-17)^2 + (y-5)^2} \leq 40$$
$$\sqrt{(x-52)^2 + (y-21)^2} \leq 40$$

Using Solver: Min total distance = 81.8 which happens at $(12.2, 21)$ (This is essentially the Akron tower. We should recommend that rather than build a new tower, Rappaport upgrade the Akron tower).