HOMEWORK #2 SOLUTIONS

Problem 0.1.

Derive the Quadratic Formula.

Proof.
The general form of a quadratic equation is $ax^2 + bx + c = 0$ where $a \neq 0$. Completing the square gives us:

$$\begin{aligned}
a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) &= -c + \frac{b^2}{4a} \\
a(x + \frac{b}{2a})^2 &= \frac{b^2 - 4ac}{4a} \\
(x + \frac{b}{2a})^2 &= \frac{b^2 - 4ac}{4a^2} \\
(x + \frac{b}{2a}) &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

□

Before answering exercise 3, we answer exercise 4.

Problem 0.2.

#1.4 *Explain why the square has the largest area among all rectangles with a given perimeter.*

Proof.
Let p be the perimeter of an arbitrary rectangle with sides x and y. Then $p = 2x + 2y$ which gives us $y = \frac{p-2x}{2}$. So the area is given by the function

$$A(x, y) = xy.$$

Substituting

$$A(x, y) = x(\frac{p - 2x}{2}) = \frac{p}{2}x - x^2.$$

This is a function in one variable, hence we may use Calc I to find a maximum:

$$\frac{d}{dx}A(x, y) = \frac{p}{2} - 2x.$$

Setting this equal to 0 and solving for x gives us

$$x = \frac{p}{4}.$$

Substituting yields

$$y = \frac{p-2x}{2} = \frac{p-2\cdot\frac{p}{4}}{2} = \frac{\frac{p}{2}}{2} = \frac{p}{4}.$$

Hence the area is maximized when $x = y = \frac{p}{4}$.

Problem 0.3.
#1.3 *Given that* $x + y = 100$*, what is the maximum value of* xy*?*

Solution.
Observe that $2(x+y) = p$. Then, by exercise #4, $x = y = \frac{200}{4} = 50$. Thus the maximum value of xy is 2500.

Problem 0.4.
#1.6 *Let* $f =$ *the value of the temperature on the Fahrenheit scale and* $c =$ *the value of the temperature on the Celsius scale. Then*

$$f = \frac{9}{5}c + 32. \tag{1}$$

Given this, when does $a)f = c$*,* $b)f = -c$*, and* $c)f = 2c$*?*

Solution.
$a)$ Substituting $c = f$ into (1)

$$c = \frac{9}{5}c + 32.$$

Multiplying by 5,

$$5c = 9c + 160. \tag{2}$$

Hence

$$-4c = 160 \Rightarrow c = -40 = f.$$

$b)$ Substituting $-c = f$ into (1) gives us in (2)

$$5c = -9c + 160.$$

So

$$14c = 160 \Rightarrow c = \frac{80}{7}.$$

Answering the question, $f = -c$ when $f = 11\frac{3}{7}$ and $c = -11\frac{3}{7}$.

c) Substituting $f = 2c$ into (1) gives us in (2)

$$10c = 9c + 160.$$

So

$$c = 160.$$

Answering the question, $f = 2c$ when $c = 160$ and $f = 320$.

Problem 0.5.
#1.7 *Give an example showing that the following is false, then change the hypothesis to make the statement true:*

$$x, y \in \mathbb{R}, x \neq 0 \neq y, \text{ and } x > y \Rightarrow -\frac{1}{x} > -\frac{1}{y}.$$

Solution.
Counterexample: Let $x = 1$ and $y = -1$. Then $x > y$ and $-\frac{1}{x} = -1 < -\frac{1}{-1} = 1$.
Correct Hypothesis: Note that if x and y have the same sign, then the operation of reciprocating flips the inequality. Hence a corrected version of our theorem is:

$$x, y \in \mathbb{R}, x \neq 0 \neq y, x > y, \text{ and } x, y \text{ have the same sign } \Rightarrow -\frac{1}{x} > -\frac{1}{y}.$$

Problem 0.6.
#1.8

Solution.

TABLE 1. Distribution of A's in Two Sections of a Calculus Course

Section	Women	A's	%	Men	A's	%	Total	A's	%
Morning	9	2	.2222	10	2	.2000	19	4	..2105
Afternoon	9	6	.6667	14	9	.6429	23	15	.6522
Course Total	18	8	.4444	24	11	.4583	42	19	.4524

The reason that the proportion of men that receive A's is higher than the proportion of women (though this is not the case in either section) is
(1) the rates vary greatly between the classes and

(2) since there are more men in the section with the higher rate of A's than men in the section with a lower rate, the higher rate carries more weight when calculating the average, whereas the women are evenly distributed and, thus, each of the lower rate and the higher rate carry the same weight when calculating the average.

Problem 0.7.
#1.9 *See text.*

Solution.
Let the stock have initial value x. Then the final value in the first scenario is

$$x - .20x + .23(x - .20x) = x - .20x + .23x - .046x = .984x.$$

The final value in the second scenario is

$$x + .20x - .18(x + .20x) = x + .20x - .18x - .036x = .984x.$$

Thus there is no net profit in either case.

Problem 0.8.
#1.13 *Let* $A := \{2k - 1 : k \in \mathbb{Z}\}$ *and* $B := \{2l + 1 : l \in \mathbb{Z}\}$. *Show* $A = B$.

Proof.
($\subseteq$) Let $a \in A$. Then there exists a $k \in \mathbb{Z}$ such that $a = 2k - 1$. Since $k \in \mathbb{Z}$ there is an $l \in \mathbb{Z}$ such that $l = k - 1$, i.e. $k = l + 1$. Substituting we have

$$a = 2k - 1 = 2(l + 1) - 1 = 2l + 1.$$

Since $l \in \mathbb{Z}$, $a \in B$ and thus $A \subseteq B$.

($\supseteq$) Let $b \in B$. Then there exists a $l \in \mathbb{Z}$ such that $a = 2l + 1$. Since $l \in \mathbb{Z}$ there is an $k \in \mathbb{Z}$ such that $k = l + 1$, i.e. $l = k - 1$. Substituting we have

$$b = 2l + 1 = 2(k - 1) - 1 = 2k - 1.$$

Since $k \in \mathbb{Z}$, $b \in A$ and thus $A \supseteq B$.

□

Problem 0.9.
#1.15 *For what conditions on the sets* A *and* B *does* $A - B = B - A$ *hold?*

Proof.
$A - B = B - A$ if and only if $A = B$.

□

Problem 0.10.
#1.17 *What are the domain and image of the absolute value function?*

Solution.
Let $f(x) = |x|$. Then the domain of f is $\mathbb{R}$ and the image of f is $\{r : r \in \mathbb{R} \text{ and } r \geq 0\}$.

Problem 0.11.
#1.18 *What real numbers exceed their reciprocals by exactly* 1*?*

Solution.
Let $R = \mathbb{R} - \{0\}$ and suppose $r \in \mathbb{R}$ such that

$$r = \frac{1}{r} + 1.$$

Since $r \neq 0$ we may multiply by r and get

$$r^2 = 1 + r.$$

This is equivalent to

$$r^2 - r - 1 = 0.$$

Solving for r

$$r = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}.$$

Problem 0.12.
#1.19 *What are the dimensions of a rectangle with perimeter* 48 *feet and area* 108 *square feet? Given $p, a \in \mathbb{R}$ with $p, a > 0$, under what conditions does there exist a rectangle with perimeter p and area a?*

Solution.
Let x and y be the sides of a rectangle. We have

$$2x + 2y = 48 \text{ and } xy = 108.$$

From the second equation we have $y = \frac{108}{x}$ and substituting into the first equation:

$$48 = 2x + 2(\frac{108}{x}).$$

Multiplying by $\frac{x}{2}$:

$$24x = x^2 + 108.$$

This is equivalent to

$$x^2 - 24x + 108 = 0.$$

Solving for x

$$x = \frac{24 \pm \sqrt{(-24)^2 - 4(1)(108)}}{2(1)} = \frac{24 \pm \sqrt{576 - 432}}{2} = \frac{24 \pm 12}{2}.$$

So $x = \frac{24+12}{2} = 18$ or $x = \frac{24-12}{2} = 6$. Notice that when $x = 18$ we have $y = 6$ and that when $x = 6$ we get $y = 18$. Hence the dimensions of the rectangle are 16 by 8.

Answering the second part of the question, let x and y be the sides of an arbitrary rectangle with perimeter p and area a. Thus $p = 2x + 2y \Leftrightarrow x + y = \frac{p}{2}$ and $a = xy$. We obtain a solution by putting $s = \frac{p}{2}$ and $a = p$ in the "Babylonian Problem" (pages $2 - 3$). Thus we have a rectangle if $(\frac{p}{2})^2 - 4a > 0$.

Problem 0.13.
#1.24 *See text.*

Solution.
The dollar is not missing. Different quantities are being compared in the last sentence of the problem:
Customers paid: $3(10 - 1) = 27$ and
Hotel personnel received: $25 + 2 = 27$.

Problem 0.14.
#1.27 *Solve* $|\frac{x}{x+1}| \leq 1$ *for* $x \in \mathbb{R}$.

Solution.
Squaring both sides gives the inequality $\frac{x^2}{(x+1)^2} \leq 1$ which is equivalent to

$$x^2 \leq x^2 + 2x + 1.$$

Canceling like the x^2 and rearranging gives us

$$2x \geq -1$$

which means

$$x \geq -\frac{1}{2}.$$

Problem 0.15.
#1.28 *Use the AGM Inequality to prove*

$$x(c - x) \tag{3}$$

is maximized when $x = \frac{c}{2}$*. Use this result to find the value of* y *that maximizes* $y(c - ay)$ *for* $a > 0$*.*

Solution.
Let us begin by reminding the reader of the AGM Inequality:

$$\sqrt{xy} \le \frac{x+y}{2}. \tag{4}$$

Of course, $x, y \in \mathbb{R}$. Note that (4) is equivalent to

$$xy \le \frac{(x+y)^2}{4}. \tag{5}$$

Putting $y = c - x$ in (3) and using (5) we have

$$x(c-x) \le \frac{(x+c-x)^2}{4} = \frac{c^2}{4}.$$

Hence the left hand side is maximized when we have equality, i.e.

$$x(c-x) = \frac{c^2}{4}.$$

Distributing the x and rewriting gives us

$$0 = x^2 - cx + \frac{c^2}{4}.$$

Multiplying by 4 yields

$$0 = 4x^2 - 4cx + c^2 = (2x-c)^2.$$

Hence $2x - c = 0$ or $x = \frac{c}{2}$.

To solve the second part of the problem, notice

$$y(c-ay) = \frac{1}{a}ay(c-ay).$$

Noting that, since $a > 0$, $\frac{1}{a}x(c-x)$ has a maximum whenever $x(c-x)$ has a maximum. Hence we put $x = ay$ and use the previous result, namely $y(c-ay)$ has a maximum when

$$x = ay = \frac{c}{2}$$

which means the maximum occurs when

$$y = \frac{c}{2a}.$$

Problem 0.16.
#1.36 *Let* $S = [3]\times[3] = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$ *and let* $T = \{(x,y) : x, y \in \mathbb{Z}$ *and* $0 \le 3x + y - 4 \le 8\}$*. Show that* $S \subseteq T$*. Does equality hold?*

Solution.
$(1,1) \in S$ and $0 \leq 3(1) + 1 - 4 = 0 \leq 8$, thus $(1,1) \in T$. As well $(3,3) \in S$ and $0 \leq 3(3) + 3 - 4 = 8 \leq 8$, thus $(3,3) \in T$. Noting that $3x + y - 4$ is an increasing function in x and y means that each of the other elements in S is also in T, hence $S \subseteq T$. Noting that $(-1,7) \in T$ shows that $S \neq T$.

Problem 0.17.
#1.40 *Show that for sets A and B,*

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

Proof.
($\subseteq$) Let $x \in (A - B) \cup (B - A)$. Then $x \in (A - B)$ or $x \in (B - A)$.
Case 1:
$x \in (A - B)$ hence $x \in A$ and $x \notin B$. Thus $x \in A \cup B$ and $x \notin A \cap B$. Which means that $x \in (A \cup B) - (A \cap B)$.
Case 2:
$x \in (B - A)$ hence $x \in B$ and $x \notin A$. Thus $x \in A \cup B$ and $x \notin A \cap B$. Which means that $x \in (A \cup B) - (A \cap B)$.
Since x is an arbitrary element and since we have considered all cases

$$(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B).$$

($\supseteq$) Let $x \in (A \cup B) - (A \cap B)$. Then $x \in (A \cup B)$ and $x \notin (A \cap B)$. This can happen in only two ways, namely either $x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$.
Case 1:
$x \in A$ and $x \notin B$. Thus $x \in A - B$ and we have $x \in (A - B) \cup (B - A)$.
Case 2:
$x \in B$ and $x \notin A$. Thus $x \in B - A$ and we have $x \in (A - B) \cup (B - A)$.
Since we have considered all possible cases, $x \in (A - B) \cup (B - A)$, i.e.

$$(A - B) \cup (B - A) \supseteq (A \cup B) - (A \cap B).$$

Because we have containment in both directions, we have

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

□